

A Converse to Banach’s Fixed Point Theorem and its CLS Completeness

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Abstract

Banach’s fixed point theorem for contraction maps has been widely used to analyze the convergence of iterative methods in non-convex problems. It is a common experience, however, that iterative maps fail to be globally contracting under the natural metric in their domain, making the applicability of Banach’s theorem limited. We explore how generally we can apply Banach’s fixed point theorem to establish the convergence of iterative methods when pairing it with carefully designed metrics.

Our first result is a strong converse of Banach’s theorem, showing that it is a *universal analysis tool* for establishing uniqueness of fixed points and for bounding the convergence rate of iterative maps to a unique fixed point. In other words, we show that, whenever an iterative map globally converges to a unique fixed point, there exists a metric under which the iterative map is contracting and which can be used to bound the number of iterations until convergence. We illustrate our approach in the widely used power method, providing a new way of bounding its convergence rate through contraction arguments.

We next consider the computational complexity of Banach’s fixed point theorem. Making the proof of our converse theorem constructive, we show that computing a fixed point whose existence is guaranteed by Banach’s fixed point theorem is CLS-complete. We thus provide the first natural complete problem for the class CLS, which was defined in [DP11] to capture the complexity of problems such as P-matrix LCP, computing KKT-points, and finding mixed Nash equilibria in congestion and network coordination games.

1 Introduction

Several widely used computational methods are fixed point iteration methods. These include gradient descent, the power iteration method, alternating optimization, the expectation-maximization algorithm, k -means clustering, and others. In several important applications, we have theoretical guarantees for the convergence of these methods. For example, convergence to a unique solution can be guaranteed when the method is explicitly, or can be related to, gradient descent on a convex function [BTN01, Nes13, BV04]. More broadly, convergence to a stationary point can be guaranteed when the method is, or can be related to, gradient descent; for some interesting recent work on the limit points of gradient descent, see [PP16, LSJR16] and their references.

Another, more general, style of analysis for proving convergence of fixed point iteration methods is via a potential (a.k.a. Lyapunov) function. For example, analyzing the power iteration method amounts to showing that, as time progresses, the unit vector maintained by the algorithm places more and more of its ℓ_2 energy on the principle eigenvector of the matrix used in the iteration, if it is unique, or, anyways, on the eigenspace spanned by the principal eigenvectors; see Appendix A.2 for a refresher. In passing, it should also be noted that the power iteration method itself is commonly used as a tool for establishing the convergence of other fixed point iteration methods, such as alternating optimization; e.g. [Har14].

Ultimately, all fixed point iteration methods aim at converging to a fixed point of their iteration map. It is, thus, unsurprising that another widely used approach for establishing convergence of these methods is by appealing to Banach’s fixed point theorem. To recall, consider an iteration map $f : \mathcal{D} \rightarrow \mathcal{D}$, and suppose that there is a distance metric d such that (\mathcal{D}, d) is a complete metric space and f is contracting according to d , i.e., for some constant $c < 1$, $d(f(x), f(y)) \leq c \cdot d(x, y)$, for all $x, y \in \mathcal{D}$. Banach’s fixed point theorem guarantees that there is a *unique* fixed point $x^* = f(x^*)$. Moreover, iterating f is bound to converge to x^* . Specifically, the t -fold composition, $f^{[t]}$, of f with itself satisfies: $d(f^{[t]}(x_0), x^*) \leq c^t d(x_0, x^*)$, for any starting point x_0 .

Given Banach’s theorem, if you established that your iteration method is contracting under some distance metric d , you would also have immediately proven that your method converges and that it may only converge to a unique point. Moreover, you can predict how many steps you need from any starting point x_0 to reach an approximate fixed point $d(f(x), x) < \epsilon$.¹ Alas, several widely used fixed point iteration methods are not generally contracting, or only contracting close to their fixed points and not the entire domain where they are defined. At least, this is typically the case for the metric d under which approximate fixed points, $d(f(x), x) < \epsilon$, are sought. There is also quite an important reason why they are not contracting: in several situations, these methods may have multiple fixed points.

Given the above motivation, our goal in this paper is to *investigate the extent to which Banach’s fixed point theorem is a universal analysis tool for establishing that a fixed point iteration method both converges and converges to a unique fixed point from any starting point*. More precisely, our question is the following: if an iterative map $f : \mathcal{D} \rightarrow \mathcal{D}$ converges to a unique fixed point x^* from any starting point, is there always a way to prove this using Banach’s fixed point theorem? Additionally, can we always use Banach’s fixed point theorem to compute how many iterations we would need to find an approximate fixed point x of f satisfying $d(x, f(x)) < \epsilon$, for some distance metric d and accuracy $\epsilon > 0$?

We study this question from both a mathematical and a computational perspective. On the mathematical side, we show a strong converse of Banach’s fixed point theorem, saying the following: given an iterative map $f : \mathcal{D} \rightarrow \mathcal{D}$, some distance metric d on \mathcal{D} , and some accuracy $\epsilon > 0$, if f has

¹Indeed, it can be easily shown that $d(f^{[t+1]}(x_0), f^{[t]}(x_0)) \leq c^t d(x_1, x_0)$. So $t = \log_{1/c} \frac{d(x_1, x_0)}{\epsilon}$ steps suffice.

a unique fixed point that the f -iteration converges to from any starting point, then for any constant $c \in (0, 1)$, there exists a distance metric d_c on \mathcal{D} such that:

1. d_c certifies uniqueness and convergence to the fixed point, by satisfying $d_c(f(x), f(y)) \leq c \cdot d_c(x, y)$, for all $x, y \in \mathcal{D}$;
2. d_c allows an analyst to predict how many iterations of f would suffice to arrive at an approximate fixed point x satisfying $d(x, f(x)) < \epsilon$; notice in particular that we are interested in finding an approximate fixed point with respect to the original distance metric d (and not the hallucinated one d_c).

Our converse theorem is formally stated as Theorem 1 in Section 3. In the same section we discuss its relationship to other known converses of Banach’s theorem known in the literature, in particular Bessaga’s and Meyers’s converse theorems. The most important improvement over these converses is that our constructed metric d_c is such that it allows us to bound the number of steps required to reach an approximate fixed point according to the metric of interest d and not the hallucinated one d_c ; namely Property 2 above holds. We discuss this implication in Section 3.3. Section 3.2 provides a sketch of our proof, and the complete details can be found in Appendix B.

Our converse implies in particular that Banach’s fixed point theorem is a universal analysis tool for establishing convergence of fixed point iteration methods with unique solutions. We evaluate the implications of our theorem by studying an important such method: the power method. The power method is a widely-used and well-understood method for computing the eigenvalues and eigenvectors of a matrix. It is well known that if a matrix A has a unique principal eigenvector, then the power method starting from a vector non-perpendicular to the principal eigenvector will converge to it. There is also a simple analysis, using a potential function argument, outlined above and in Appendix A.2, pinning down the rates of convergence.

Our converse to Banach’s theorem, guarantees that, besides the potential function argument, there must also exist a distance metric under which the iteration employed by the power method is a contraction map. Such a distance metric is not easy to identify, as contraction under ℓ_p -norms fails; we provide counter-examples in Section 4. Following the ideas in the proof of our converse theorem, we are able to identify a distance metric under which the power method is contracting, while at the same time it remains tightly related to the standard norms close to the principal eigenvector. See Lemma 1. By satisfying both Desiderata 1 and 2 above, our distance metric serves as an alternative proof for establishing that the power method converges and for pinning down its convergence rates. See Corollary 3.

We close the circle by studying Banach’s fixed point theorem from a computational standpoint. Recent work of Daskalakis and Papadimitriou [DP11] has identified a complexity class, CLS, where finding a Banach fixed point lies. CLS, defined formally in Section 5, is a complexity class at the intersection of PLS [JPY88] and PPAD [Pap94]. Roughly speaking, PLS contains total problems whose existence of solutions is guaranteed by a potential function argument, while PPAD contains total problems whose existence of solutions is guaranteed by Brouwer’s fixed point theorem. CLS, lying in the intersection of PLS and PPAD, contains computational problems whose existence of solutions is guaranteed by both a potential function and a fixed point argument.²

Unsurprisingly CLS contains several interesting problems, whose complexity is not known to lie in P, but which also are unlikely to be complete for PPAD or PLS. One of these problems is finding a Banach fixed point itself. Others include the P-matrix Linear Complementarity Problem, finding

²More precisely, it contains all problems reducible to CONTINUOUS LOCALOPT, defined in Section 5, and which doesn’t necessarily capture the whole intersection of PPAD and PLS.

mixed Nash equilibria of network coordination and congestion games, computational problems related to finding KKT points, and solving Simple Stochastic Games; see [DP11] for precise definitions of these problems and for references. Moreover, recent work has provided cryptographic hardness results for CLS [HY17] based on obfuscation, improving upon work which proved cryptographic hardness results for PPAD [BPR15].

Ultimately, the definition of CLS was inspired by a vast range of total problems that could not be properly classified as complete in PPAD or PLS due to the nature of their totality arguments. However, no natural complete problem for this class has been identified, besides CONTINUOUS LOCALOPT, through which the class was defined. By making our converse to Banach’s fixed point theorem constructive, we show that finding a Banach fixed point is CLS-complete. More precisely, in Section 5 we define problem METRICBANACH, whose input is a function f and a function d , and whose goal is to either output an approximate fixed point of f , a violation of the continuity of f or d , a violation of the contraction of f with respect to d , or a violation by d of any of the metric properties. This problem is syntactically total, and is shown to be CLS-complete in Theorem 2. Additionally, we establish the CLS-completeness of the promise problem BANACH, which is similar to METRICBANACH except we also have the promise that d is a distance metric, and that the domain of f is a complete metric space with respect to d .³ This is shown in Theorem 3.

Lastly, it is worth pointing out that, while some problems in CLS (e.g. Banach fixed points, simple stochastic games) have unique solutions, most do not. Given that contraction maps have unique fixed points, the way we bypass the potential oxymoron, is by accepting as solutions violations of continuity, metric properties, or contraction.

We note that contemporaneously and independently from our work, Fearnley et al. [FGMS17] have also identified a CLS-complete problem related to Banach’s fixed point theorem. Their problem, called METAMETRICCONTRACTION, takes as input a function f and a metametric d , and asks to find an approximate fixed point of f , a violation of the continuity of either f or d , or a violation of the contraction of f with respect to d . In comparison to our CLS-completeness results, the CLS-hardness of BANACH in our paper is stronger than that of METAMETRICCONTRACTION as the input to BANACH is promised to be a metric. On the other hand, the containment of METAMETRICCONTRACTION into CLS is stronger than the containment of BANACH, as the containment holds even if the input is a meta-metric. As BANACH is polynomial-time reducible to METRICBANACH, the containment of METRICBANACH in CLS is a stronger statement than the containment of BANACH in CLS. The same way the containment of METAMETRICCONTRACTION in CLS is a stronger statement than the containment of METRICBANACH in CLS as METRICBANACH is polynomial-time reducible to METAMETRICCONTRACTION.

2 Notation and Preliminaries

Basic Notation We use \mathbb{R}_+ to refer to set of non-negative real numbers and \mathbb{N}_1 is the set of natural numbers except 0. We call a function f *selfmap* if it maps a domain \mathcal{D} to itself, i.e. $f : \mathcal{D} \rightarrow \mathcal{D}$. For a selfmap f we use $f^{[n]}$ to refer to the n times composition f with it self, i.e. $\underbrace{f(f(\dots f(\cdot)))}_{n \text{ times}}$.

We use $\|\cdot\|_p$ to refer to the ℓ_p norm of a vector in \mathbb{R}^n . We use \mathcal{D}/\sim to refer to the set of equivalence classes of the equivalence relation \sim on a set \mathcal{D} . Finally, we use S^* to refer to the Kleene star of a set S .

³We note that, if f is not defined on a complete metric space, then there is no guarantee that the iterative application of f will converge to a fixed point. In this sense, BANACH is closer than METRICBANACH to capturing the computational complexity of fixed points whose existence is guaranteed by Banach’s theorem.

A real valued function $g : \mathcal{D}^2 \rightarrow \mathbb{R}$ is called *symmetric* if $g(x, y) = g(y, x)$ and *anti-symmetric* if $g(x, y) = -g(y, x)$.

In Appendix A.1, the reader can find some well-known definitions that we are going to use in the rest of the paper. More precisely in the field of:

Topological Spaces, we define the notion of: topology, topological spaces, open sets, closed sets, interior of a set A , denoted $\text{Int}(A)$, closure of a set A , denoted $\text{Clos}(A)$.

Metric Spaces, we define the notion of: distance metric, metric space, diameter, bounded metric space, continuous function, open and closed sets in a metric space, compact set, locally compact metric space, proper metric space, open and closed balls, Cauchy sequence, complete metric space, equivalent metrics, continuity, Lipschitz continuity, contraction property, fixed point.

Because of its importance for the rest of the paper we also give here the definition of a distance metric and metric space.

Definition 1. Let \mathcal{D} be a set and $d : \mathcal{D}^2 \rightarrow \mathbb{R}$ a function with the following properties:

- (i) $d(x, y) \geq 0$ for all $x, y \in \mathcal{D}$.
- (ii) $d(x, y) = 0$ if and only if $x = y$.
- (iii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{D}$.
- (iv) $d(x, y) \leq d(x, z) + d(z, x)$ for all $x, y, z \in \mathcal{D}$. This is called *triangle inequality*.

Then we say that d is a *metric* on \mathcal{D} , and (\mathcal{D}, d) is a *metric space*.

Basic Iterative Procedure. If a selfmap f has a fixed point and is continuous, we can define the following sequence of points $x_{n+1} = f(x_n)$ where the starting point x_0 can be picked arbitrarily. If (x_n) converges to a point \bar{x} then

$$\lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) \Rightarrow \lim_{n \rightarrow \infty} x_{n+1} = f\left(\lim_{n \rightarrow \infty} x_n\right) \Rightarrow \bar{x} = f(\bar{x}).$$

This observation implies that a candidate procedure for computing a fixed point of a selfmap f is to *iteratively* apply the function f starting from an arbitrary point x_0 . If this procedure converges then the limit is a fixed point x^* of f . We will refer to this method of computing fixed points as the *Basic Iterative Procedure*.

Arithmetic Circuits. In Section 5 we work with functions from continuous domains to continuous domains represented as *arithmetic circuits*. An arithmetic circuit is defined by a directed acyclic graph (DAG). The inputs to the circuit are in-degree 0 nodes, and the outputs are out-degree 0 nodes. Each non-input node is a gate from the set $\{+, -, *, \max, \min, >\}$, performing an operation on the outputs of its in-neighbors. The meaning of the “ $>$ ” gate is $>(x, y) = 1$ if $x > y$ and 0 otherwise. We also allow “output a rational constant” gates. These are gates without any inputs, which output a rational constant.

3 Converse Banach Fixed Point Theorems

We start, in Section 3.1, with an overview of known converses to Banach’s fixed point theorem. We also explain why these converses are not enough to prove that Banach’s fixed point theorem is a universal tool for analyzing the convergence of iterative algorithms. Then, in Section 3.2, we prove a stronger converse theorem that demonstrates the universality of Banach’s fixed point theorem for the analysis of iterative algorithms. Before beginning, we formally state Banach’s fixed Point Theorem. A useful survey of the applications of this theorem can be found in [Con14].

Banach's Fixed Point Theorem. Suppose d is a distance metric function such that (\mathcal{D}, d) is a complete metric space, and suppose that $f : \mathcal{D} \rightarrow \mathcal{D}$ is a contraction map according to d , i.e.

$$d(f(x), f(y)) \leq c \cdot d(x, y), \forall x, y, \text{ for some } c < 1. \quad (1)$$

Then f has a unique fixed point x^* and the convergence rate of the Basic Iterative Procedure with respect to d is c . That is, $d(f^{[n]}(x_0), x^*) < c^n \cdot d(x_0, x^*)$, for all x_0 .

3.1 Known Converses to Banach's Fixed Point Theorem

The first known converse to Banach's fixed point theorem is the following [Bes59].

Bessaga's Converse Theorem. Let f be a map from \mathcal{D} to itself, and suppose that $f^{[n]}$ has unique fixed point for every $n \in \mathbb{N}_1$. Then, for every constant $c \in (0, 1)$, there exists a distance metric d_c such that (\mathcal{D}, d_c) is a complete metric space and f is a contraction map with respect to d_c with contraction constant c .

The implication of the above theorem is that, if we want to prove existence and uniqueness of fixed points of $f^{[n]}$ for all n , then Banach's fixed point theorem is a universal way to do it. Moreover, there is a potential function of the form $p(x) = d_c(x, f(x))$, where d_c is a distance metric, that decreases under successive applications of f , and successive applications of f starting from any point x_0 are bound to converge to the unique fixed point of f .

Unfortunately, d_c cannot provide any information about the number of steps that the Basic Iterative Procedure needs before computing an approximate fixed point under some metric d of interest. The reason is that, after $\log_c \varepsilon$ steps of the Basic Iterative Procedure, we only have $d_c(x_n, f(x_n)) \leq \varepsilon$. However, d_c might not have any relation to d , hence an approximate fixed point under d_c may not be one for d . So Bessaga's theorem is not useful for bounding the running time of iterative methods for approximate fixed point computation.

Given the above discussion, it is reasonable to expect that a converse to Banach's theorem that is useful for bounding the running time of approximate fixed point computation methods should take into account, besides the function f and its domain \mathcal{D} , the distance metric d under which we are interested in computing approximate fixed points. One step in this direction has already been made by Meyers [Mey67].

Meyers's Converse Theorem. Let (\mathcal{D}, d) be a complete metric space, where \mathcal{D} is compact, and suppose that $f : \mathcal{D} \rightarrow \mathcal{D}$ is continuous with respect to d . Suppose further that f has a unique fixed point x^* , that the Basic Iterative Method converges to x^* from any starting point, and that there exists an open neighborhood U of x^* such that $f^{[n]}(U) \rightarrow \{x^*\}$. Then, for any $c \in (0, 1)$, there exists a distance metric d_c , which is topologically equivalent to d , such that (\mathcal{D}, d_c) is a complete metric space and f is a contraction map with respect to d_c with contraction c .

Compared to Bessaga's theorem, the improvement offered by Meyer's Theorem is that, instead of the existence of an arbitrary metric, it proves the existence of a metric that is topologically equivalent to the metric d . However, this is still not enough to bound the number of steps needed by the Basic Iterative Procedure in order to arrive at a point x_n such that $d(x_n, f(x_n)) \leq \varepsilon$. Our goal in the next section is to close this gap. We will also replace the compactness assumption with the assumption that (\mathcal{D}, d) is proper, so that the converse holds for unbounded spaces.

3.2 A New Converse to Banach's Fixed Point Theorem

The main technical idea behind our converse to Banach's fixed point theorem is to adapt the proof of Meyers's theorem to get a distance metric d_c with the property $d_c(x, y) \geq d(x, y)$ everywhere, except maybe for the region $d(x, x^*) \leq \varepsilon$. This implies that, if we guarantee that $d_c(x_n, x^*) \leq \varepsilon$, then $d(x_n, x^*) \leq \varepsilon$.

Theorem 1. Suppose (\mathcal{D}, d) is a complete, proper metric space, $f : \mathcal{D} \rightarrow \mathcal{D}$ is continuous with respect to d and the following hold:

1. f has a unique fixed point x^* ;
2. for every $x \in \mathcal{D}$, the sequence $(f^{[n]}(x))$ converges to x^* with respect to d ; moreover there exists an open neighborhood U of x^* such that $f^{[n]}(U) \rightarrow \{x^*\}$.

Then, for every $c \in (0, 1)$ and $\varepsilon > 0$, there exists a distance metric function $d_{c,\varepsilon}$ that is topologically equivalent to d and is such that $(\mathcal{D}, d_{c,\varepsilon})$ is a complete metric space and

$$\forall x, y \in \mathcal{D} : d_{c,\varepsilon}(f(x), f(y)) \leq c \cdot d_{c,\varepsilon}(x, y); \quad (2a)$$

$$\forall x, y \in \mathcal{D} : d_{c,\varepsilon}(x, y) \leq \varepsilon \implies \min\{d(x^*, x), d(x^*, y), d(x, y)\} \leq 2\varepsilon. \quad (2b)$$

Remark. Notice that the continuity of f is a necessary assumption for the above statement to hold, as (2a) implies continuity given that $d_{c,\varepsilon}$ and d are topologically equivalent. Also the condition 2. of the theorem is implied by the existence of $d_{c,\varepsilon}$ and it is not true even if $f^{[n]}(x) \rightarrow x^*$ for any $n \in \mathbb{N}$, since counter examples exist. Therefore this assumption is also necessary for our theorem to hold.

As we already discussed, our proof adapts Meyers's proof in a careful manner so that (2b) is satisfied. We give here a proof sketch postponing the complete details to Appendix B, where we repeat also all the technical details proven by Meyers [Mey67].

Proof Sketch. The construction of d_c follows three steps:

- I. We first construct a metric d_M , which is topologically equivalent to d , and with respect to which f is non-expanding. It also holds that $d_M(x, y) \geq d(x, y)$ for all $x, y \in \mathcal{D}$ and therefore Property (2b) can be transferred from d_M to d .
- II. Given d_M , we proceed to construct a “distance” function $\rho_{c,\varepsilon}$, which satisfies (2a) and all the metric properties except maybe for the triangle inequality. Moreover $\rho_{c,\varepsilon}$ satisfies that $\rho_{c,\varepsilon}(x, y) \geq d_M(x, y)$ if $\max\{d(x^*, x), d(x^*, y)\} \geq \varepsilon$, and therefore (2b) is preserved.
- III. Given $\rho_{c,\varepsilon}$, we construct the sought after metric $d_{c,\varepsilon}$ by taking it equal to the $\rho_{c,\varepsilon}$ -geodesic distance. Given the properties of $\rho_{c,\varepsilon}$ and the definition of $d_{c,\varepsilon}$, we can prove that $d_{c,\varepsilon}$ is a metric and Properties (2a) and (2b) hold.

□

3.3 Corollaries of Theorem 1

Property (2b) of the metric output by Theorem 1 has some interesting corollaries that we would not be able to get using the known converses to Banach's theorem discussed in Section 3.1. The first one is that we can now compute, from $d_{c,\varepsilon}$, the number of iterations needed in order to get to within ε of the fixed point x^* of f from any starting point $x_0 \in \mathcal{D}$.

Corollary 1. Under the assumptions of Theorem 1, starting from a point $x_0 \in \mathcal{D}$, and for any constant $c \in (0, 1)$, the Basic Iterative Procedure finds a point x such that $d(x, x^*) \leq \varepsilon$ after

$$\frac{\log(d_{c,\varepsilon/2}(x_0, f(x_0))) + \log((2 - 2c)/\varepsilon)}{\log(1/c)}$$

iterations, where $d_{c,\varepsilon/2}$ is the metric guaranteed by Theorem 1.

Proof. Let $d_{c,\varepsilon/2}$ be the distance metric guaranteed by Theorem 1 with parameters $c, \varepsilon/2$. Let also (x_n) be the sequence produced by the Basic Iterative Procedure. Since f is a contraction with respect

to $d_{c,\varepsilon/2}$, we have $d_{c,\varepsilon/2}(x_n, x^*) \leq \frac{c^n}{1-c} d_{c,\varepsilon/2}(x_0, x_1)$. If we make sure that $d_{c,\varepsilon/2}(x_n, x_{n+1}) \leq \varepsilon/2$ then according to Theorem 1 $d(x_n, x^*) \leq \varepsilon$. So the number of steps that are needed are:

$$\frac{c^n}{1-c} d_{c,\varepsilon/2}(x_0, x_1) \leq \frac{\varepsilon}{2} \Leftrightarrow n \geq \frac{\log(d_{c,\varepsilon/2}(x_0, x_1)) + \log((2-2c)/\varepsilon)}{\log(1/c)}.$$

□

In Corollary 1, for any given ε of interest, we have to identify a different distance metric $d_{c,\varepsilon/2}$, guaranteed by Theorem 1, to bound the number of steps required by the Basic Iterative Procedure to get to within ε from the fixed point. Sometimes we are interested in the explicit tradeoff between the number of steps required to get to the proximity of the fixed point and the amount of proximity ε . To find such a tradeoff we have to make additional assumptions on f . A mild assumption that is commonly satisfied by iterative procedures for non-convex problems is that the Basic Iterative Procedure *locally converges* to the fixed point x^* . That is, if x_0 is appropriately close to x^* , then the Basic Iterative Procedure converges. A common way of proving local convergence is to prove that f is a contraction with respect to d *locally* for $x, y \in \bar{B}(x^*, \varepsilon)$. Theorem 1 provides a way to extend this local contraction property to the whole domain \mathcal{D} and get an explicit closed form of the tradeoff between the number of steps and ε , as implied by the following result.

Corollary 2. *Under the assumptions of Theorem 1, and the assumption that there exists $0 < c < 1$, $\delta > 0$ such that*

$$d(f(x), f(y)) \leq c \cdot d(x, y) \text{ for all } x, y \in \bar{B}(x^*, \delta),$$

starting from any point $x_0 \in \mathcal{D}$, the Basic Iterative Procedure finds a point x such that $d(x, x^) \leq \varepsilon$ after*

$$\frac{\log(d_{c,\delta/2}(x_0, f(x_0))) + \log(1/\varepsilon) + \log(1-c) + 1}{\log(1/c)} + 1$$

iterations, where $d_{c,\delta/2}$ is the metric guaranteed by Theorem 1.

Proof. Using Corollary 1, we get that after $n = \frac{\log(d_{c,\delta/2}(x_0, f(x_0))) + \log((2-2c)/\delta)}{\log(1/c)}$ iterations we will have $d(x_n, x^*) \leq \delta$ or $d(x_{n+1}, x^*) \leq \delta$. Since in $\bar{B}(x^*, \delta)$, f is a contraction with respect to d , it certainly must be that $d(x_{n+1}, x^*) \leq \delta$. By the same token, $d(x_{n+1+m}, x^*) \leq c^m d(x_{n+1}, x^*)$, for all $m > 0$. Therefore, to guarantee $d(x_{n+1+m}, x^*) \leq \varepsilon$, it suffices to take $m \geq \frac{-\log(1/\delta) + \log(1/\varepsilon)}{\log(1/c)}$. So in total we need $n + 1 + m$ iterations, implying the number of iterations stated in the statement of the corollary. □

4 Example: The Power Method as a Contraction Map

In this section we show how we can apply the ideas of the previous section to the analysis of the widely-used *power method* for computing eigenvalues and eigenvectors of a matrix. The power method is well-understood both from a theoretical and from a practical point of view. On the theoretical side, it is well-known that the power method converges and the tradeoff between the number of steps and its accuracy is also well-understood. When intuiting about why the power method works, it is natural to expect that its iteration (as a map f from vectors to vectors) contracts the space towards the fixed point, which is the principal eigenvector.⁴ Still, to the best of our knowledge, all known arguments for establishing the convergence of the power method use a

⁴Let us assume that it is unique, and for our discussion here let us restrict the domain of f to be all vectors non perpendicular to the principal eigenvector. Starting from any vector in this set, it is well-known that the power method converges to the principal eigenvector.

potential function argument, without establishing the contraction intuition that we just described. Still, our converse theorem guarantees that there is a distance metric under which the iteration of the power method contracts. We want to identify such a distance metric here.

The difficulty arises when we try to prove the contraction property with distance metric $d(x, y) = \|x - y\|_p$, as we quickly run into counterexamples; see below. What Theorem 1 suggests is that we should look for an arbitrary metric that makes the power method a contraction map, while only remaining tightly related to $\|x - y\|_p$ close to the fixed point. This is our goal in this section. We start with some counterexamples why the power method is not a contraction with respect to $\|x - y\|_p$ and we proceed to construct a distance metric that makes the power method a contraction map, also satisfying (2b) of Theorem 1. Note that the existence of such a metric is guaranteed by Theorem 1.

For a detailed description of the power method we refer to the survey of [Pan11]. We give a small overview in Appendix A.2. We study the following setting: Let $A \in \mathbb{R}^{n \times n}$, $\{q_i\}$ be the set of unit eigenvectors of A that forms a basis of \mathbb{R}^n , and suppose that the corresponding set of real eigenvalues $\{\lambda_i\}$ is such that $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$. Let v_0 be an arbitrary initial vector, not perpendicular to q_1 , with $\|v_0\| = 1$. We can write v_0 as a linear combination of the eigenvectors of A , that is for some $c_1, \dots, c_n \in \mathbb{R}$ it holds that $v_0 = c_1 q_1 + c_2 q_2 + \dots + c_n q_n$ and since we assumed that v_0 is not perpendicular to q_1 we have that $c_1 \neq 0$. Therefore each iteration of the power method applies the following map

$$(c_1, c_2, \dots, c_n) \mapsto \frac{(\lambda_1 c_1, \lambda_2 c_2, \dots, \lambda_n c_n)}{\|(\lambda_1 c_1, \lambda_2 c_2, \dots, \lambda_n c_n)\|}.$$

In Appendix A.2 we provide the standard analysis of the convergence of the power method to the principal eigenvector q_1 , and bound the rate of convergence, using the standard potential function argument. Here we re-prove the same properties by establishing that the power method is a contraction map under the right distance metric, which we construct.

Counterexamples for $\|\cdot\|_p$. We will show a counter example for the ℓ_1 norm but it is very easy to find similar counter examples for every ℓ_p norm. Let $n = 2$, $\lambda_1 = 2$, $\lambda_2 = 1$. The power iteration is $f(x, y) = \frac{(2x, y)}{|2x| + |y|}$. We set $(x_1, y_1) = (1/3, 2/3)$ and $(x_2, y_2) = (1/4, 3/4)$. We get that $\|f(x_1, y_1) - f(x_2, y_2)\|_1 = \|(1/2, 1/2) - (2/5, 3/5)\|_1 = 2/10$. Also $\|(x_1, y_1) - (x_2, y_2)\|_1 = \|(1/3, 2/3) - (1/4, 3/4)\|_1 = 2/12$ and therefore $\|f(x_1, y_1) - f(x_2, y_2)\|_1 > \|(x_1, y_1) - (x_2, y_2)\|_1$.

Definition of the Appropriate Metric. We define the following metric with respect to two vectors $v = (v_1, \dots, v_n)^T$ and $u = (u_1, \dots, u_n)^T$

$$d_1(v, u) = \sum_{j=2}^n \left| \frac{v_j}{v_1} - \frac{u_j}{u_1} \right|$$

it is then easy to see the following

Lemma 1. *The power method is a contraction with respect to d_1 , with contraction constant $c = \lambda_2/\lambda_1$.*

Proof. Let f be the iteration of the power method, we have that

$$\begin{aligned} d_1(f(v), f(u)) &= \sum_{j=2}^n \left| \frac{\lambda_j v_j}{\lambda_1 v_1} - \frac{\lambda_j u_j}{\lambda_1 u_1} \right| = \sum_{j=2}^n \frac{\lambda_j}{\lambda_1} \left| \frac{v_j}{v_1} - \frac{u_j}{u_1} \right| \leq \frac{\lambda_2}{\lambda_1} \sum_{j=2}^n \left| \frac{v_j}{v_1} - \frac{u_j}{u_1} \right| \Rightarrow \\ d_1(f(v), f(u)) &\leq \frac{\lambda_2}{\lambda_1} d_1(v, u) \end{aligned}$$

Therefore the lemma holds. □

But following the discussions that we had in the previous section, d_1 is not a metric that we care for. Such a metric is the norm of the space $\|\cdot\|_1$. We also observe that we can without loss of generality assume that $\|v_0\|_1 = 1$ since we can do a normalization at each step. Among the vector with unit ℓ_1 norm the only fixed point obviously is the $(1, 0, \dots, 0)^T$. Now let's consider the case where after k iterations $d_1(u = v_k, e_1) \leq \varepsilon$, then we have

$$\sum_{j=2}^n \left| \frac{u_j}{u_1} \right| \leq \varepsilon \Rightarrow \sum_{j=2}^n |u_j| \leq \varepsilon \cdot |u_1| \leq \varepsilon$$

But since $\|u\|_1 = 1$ and since $u_1 > 0$ without loss of generality we also get $|1 - u_1| \leq \varepsilon$. Therefore

$$\|u - e_1\| = |1 - u_1| + \sum_{j=2}^n |u_j| \leq 2\varepsilon$$

Using these observations and following the same approach as in Corollaries 1-2 we get that:

Corollary 3. *Starting from a vector v_0 not perpendicular to q_1 , the power method finds a vector u , such that $\|u - e_1\|_1 \leq \varepsilon$ after*

$$\frac{\log(d_1(v_0, e_1)) + \log(2/\varepsilon)}{\log(\lambda_1/\lambda_2)} \text{ number of iterations.}$$

5 Banach is Complete for CLS

As discussed in Section 1, the complexity class CLS was defined in [DP11] to capture problems in the intersection of PPAD and PLS, such as P-matrix LCP, mixed Nash equilibria of congestion and multi-player coordination games, finding KKT points, etc. It also contains computational variants of finding fixed points whose existence is guaranteed by Banach's fixed point theorem. In this section, we close the circle by proposing two variants of Banach fixed point computation that are both CLS-complete. Our CLS completeness results are obtained by making our proof of Theorem 1 constructive. We start with a formal definition of CLS, which is defined in terms of the problem CONTINUOUS LOCALOPT.

Definition 2. CONTINUOUS LOCALOPT takes as input two functions $f : [0, 1]^3 \rightarrow [0, 1]^3$, $p : [0, 1]^3 \rightarrow [0, 1]$, both represented as arithmetic circuits, and two rational positive constants ε and λ . The desired output is any of the following:

- (CO1) a point $x \in [0, 1]^3$ such that $p(f(x)) \geq p(x) - \varepsilon$.
- (CO2) two points $x, x' \in [0, 1]^3$ violating the λ -Lipschitz continuity of f , i.e.
 $|f(x) - f(x')| > \lambda|x - x'|$.
- (CO3) two points x, x' violating the λ -Lipschitz continuity of p , i.e.
 $|p(x) - p(x')| > \lambda|x - x'|$.

The class CLS is the set of search problems that can be reduced to CONTINUOUS LOCALOPT.

The variant of Banach's theorem that is known to belong to CLS is CONTRACTION MAP, a problem defined in [DP11]. CONTRACTION MAP targets fixed points whose existence is guaranteed by Banach's fixed point theorem when f is a contraction map with respect to the ℓ_p metric of $[0, 1]^3$. However, it doesn't capture the generality of Banach's theorem, since the latter can be applied to any complete metric space.

In this section we consider the general problem of computing a fixed point whose existence is guaranteed by Banach's fixed point theorem. Because we aim to work with arbitrary distance metrics, we assume that the distance metric is given as an input in the form of an arithmetic circuit. We call this problem METRICBANACH.

Definition 3. METRICBANACH takes as input two functions $f : [0, 1]^3 \rightarrow [0, 1]^3$, $d : [0, 1]^3 \times [0, 1]^3 \rightarrow \mathbb{R}$, both represented as arithmetic circuits, and three rational positive constants ε , λ , $c < 1$. The desired output is any of the following:

- (Oa) a point $x \in [0, 1]^3$ such that $d(x, f(x)) \leq \varepsilon$
- (Ob) two points $x, x' \in [0, 1]^3$ disproving the contraction of f w.r.t. d with constant c , i.e.
 $d(f(x), f(x')) > c \cdot d(x, x')$
- (Oc) two points $x, x' \in [0, 1]^3$ disproving the λ -Lipschitz continuity of f , i.e.
 $|f(x) - f(x')| > \lambda |x - x'|$.
- (Od) four points $x_1, x_2, y_1, y_2 \in [0, 1]^3$ with $x_1 \neq x_2$ and $y_1 \neq y_2$ disproving the λ -Lipschitz continuity of $d(\cdot, \cdot)$, i.e. $|d(x_1, x_2) - d(y_1, y_2)| > \lambda (|x_1 - y_1| + |x_2 - y_2|)$.
- (Oe) points $x, y, z \in [0, 1]^3$ violating any of the metric properties of d ((i)-(iv) of Definition 1).

Theorem 2. METRICBANACH is CLS-complete.

We present the proof of Theorem 2 in Section 5.1. If we carefully inspect the proof of Banach's theorem we see that the existence of an exact fixed point x^* is guaranteed only when the metric space is complete. The reason is that without completeness the sequence of approximate fixed points, which can be shown to be a Cauchy sequence, might not converge. Therefore to capture the fixed points whose existence is guaranteed by Banach's theorem it is important that the metric space $([0, 1]^3, d)$ be complete. Unlike the metric and the contraction properties, the completeness is not a property that could be defined in a syntactic way. For this reason we add the completeness property in a semantic way. We are still able to show that even this semantic version of Banach's theorem is complete for CLS.

Definition 4. The (promise problem) BANACH has the same input and output as the problem METRICBANACH, but we also have the promise that d satisfies the properties of a distance metric (Definition 1) and the promise that $([0, 1]^3, d)$ is a complete metric space.

Theorem 3. BANACH is CLS-complete.

5.1 Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2. We first show that METRICBANACH belongs to CLS even when we disallow (Oe). Starting from an instance $(f, d, \varepsilon, \lambda, c)$ of CONTINUOUS LOCALOPT we create the following instance

$$(f'(x) = f(x), p(x) = d(x, f(x)), \varepsilon' = (1 - c) \cdot \varepsilon, \lambda' = \lambda)$$

Now we have to show that any output of the CONTINUOUS LOCALOPT with input $(f, p, \varepsilon', \lambda)$ will give us a output of METRICBANACH with input $(f, d, \varepsilon, \lambda, c)$.

(CO1) \implies If $d(f(x), f(f(x))) > c \cdot d(x, f(x))$ then $(x, f(x))$ satisfies (Ob). Otherwise

$$\begin{aligned} p(f(x)) &\geq p(x) - \varepsilon' \Rightarrow d(f(x), f(f(x))) \geq d(x, f(x)) - \varepsilon' \Rightarrow \\ c \cdot d(x, f(x)) &\geq d(f(x), f(f(x))) \geq d(x, f(x)) - \varepsilon' \Rightarrow \\ c \cdot d(x, f(x)) &\geq d(x, f(x)) - (1 - c) \cdot \varepsilon \Rightarrow \\ (1 - c) \cdot d(x, f(x)) &\leq (1 - c) \cdot \varepsilon \Rightarrow \\ d(x, f(x)) &\leq \varepsilon \end{aligned}$$

Therefore x satisfies (Oa) and therefore is a solution of METRICBANACH.

(CO2) \implies (Oc).

(CO3) \implies Without loss of generality let $\|x - f(x)\| \leq \|y - f(y)\|$. If $x = f(x)$ then if $d(x, f(x)) = 0$ we immediately satisfy (Oa) otherwise we satisfy (Oe). Otherwise we can give $x_1 = x$, $x_2 = f(x)$, $y_1 = y$, $y_2 = f(y)$ and since $x_1 \neq x_2$, $y_1 \neq y_2$ we satisfy (ii) of (Od).

This implies that any output of CONTINUOUS LOCALOPT at the instance $(f', p, \varepsilon', \lambda')$ can produce an output to the instance $(f, d, \varepsilon, \lambda, c)$ of the METRICBANACH problem. Therefore METRICBANACH \in CLS.

Now we are going to show the opposite direction and reduce CONTINUOUS LOCALOPT to METRICBANACH. Starting from an instance $(f, p, \varepsilon, \lambda)$ of CONTINUOUS LOCALOPT we define for any $x, y \in [0, 1]^3$,

$$\kappa(x, y) = \min \left\{ -\frac{p(x)}{\varepsilon}, -\frac{p(y)}{\varepsilon} \right\}$$

We also remind the reader the definition of the *discrete metric*

$$d_S(x, y) = 1 \text{ if } x \neq y \text{ and } d_S(x, x) = 0$$

Finally we define the *smooth interpolation function* for $w \leq 0$

$$B(w) = (1 - (\lceil w \rceil - w))c^{\lceil w \rceil} + ((\lceil w \rceil - w))c^{\lceil w \rceil + 1}$$

The basic observation about $B(\cdot)$ since $c < 1$ is that $c^{\lceil \kappa(x, y) \rceil + 1} \leq B(\kappa(x, y)) \leq c^{\lceil \kappa(x, y) \rceil}$.

Based on these definitions we create the following instance of METRICBANACH

$$f' = f, d(x, y) = B(\kappa(x, y)) \cdot d_S(x, y), \varepsilon' = \frac{1}{c}, \lambda' = \max \left\{ \lambda, \left\lceil c^{-1/\varepsilon} \lambda \frac{\ln(1/c)}{\varepsilon} \right\rceil \right\}, c = 1 - 0.1\varepsilon$$

As in the previous reduction we have to show that any result of the METRICBANACH with input $(f, d, \varepsilon', \lambda, c)$ will give us a result of CONTINUOUS LOCALOPT with input $(f, p, \varepsilon, \lambda)$.

(Oa) \implies If $p(f(x)) \geq p(x)$ then x satisfies (CO1). Otherwise we can see that $\kappa(x, f(x)) = -p(x)/2\varepsilon$ and $x \neq f(x)$ so

$$\begin{aligned} d(x, f(x)) \leq \varepsilon' &\Rightarrow B(\kappa(x, y)) \leq \varepsilon' \Rightarrow \left(\frac{p(x)}{\varepsilon} \right) \log(1/c) \leq \log(\varepsilon') \Rightarrow \frac{p(x)}{\varepsilon} \leq \frac{\log(\varepsilon')}{\log(1/c)} \\ &\Rightarrow p(x) \leq \varepsilon \end{aligned}$$

so $p(f(x)) \geq 0 \geq p(x) - \varepsilon$ and so x satisfies (CO1).

(Ob) \implies As in the previous case we may assume that $p(f(x)) \leq p(x) - \varepsilon$ and that $p(f(y)) \leq p(y) - \varepsilon$. Without loss of generality we can assume that $p(x) > p(y)$. If also $p(f(x)) \geq p(f(y))$ then $\kappa(x, y) = -p(x)/\varepsilon$ and $\kappa(f(x), f(y)) = -p(f(x))/\varepsilon$. Therefore

$$d(x, y) = B(\kappa(x, y)), d(f(x), f(y)) = B(\kappa(f(x), f(y)))$$

Now if (Ob) is satisfied then

$$\begin{aligned} c^{-\left\lfloor \frac{p(f(x))}{\varepsilon} \right\rfloor} &\geq d(f(x), f(y)) = B(\kappa(f(x), f(y))) > c \cdot B(\kappa(x, y)) = c \cdot d(x, y) \geq c^2 \cdot c^{-\left\lfloor \frac{p(x)}{\varepsilon} \right\rfloor} \\ &\implies \left\lfloor \frac{p(f(x))}{\varepsilon} \right\rfloor \geq \left\lfloor \frac{p(x)}{\varepsilon} \right\rfloor - 2 \Rightarrow p(f(x)) \geq p(x) - 2\varepsilon \quad ^5 \end{aligned}$$

⁵At this point we should have set $\kappa(x, y) = \min\{-2p(x)/\varepsilon, 2p(y)/\varepsilon\}$ to get the inequality $p(f(x)) \geq p(x) - \varepsilon$ but this would complicate the calculations in the rest of the cases. It is clear though that we could scale every parameter so that ε becomes 2ε and nothing changes.

Therefore x satisfies (CO1).

Now similarly if $p(f(y)) > p(f(x))$ then $p(f(y)) > p(x) - \varepsilon$. But by our assumption that $p(x) > p(y)$ we get $p(f(y)) > p(y) - \varepsilon$. Therefore y satisfies (CO1).

(Oc) \implies (CO2).

(Od) \implies We will analyze the function $h(x) = c^{-x}$ when $x \in [0, 1/\varepsilon]$. By the mean value theorem we have that the Lipschitz constant ℓ_h of h is less than $\max_{x \in [0, 1/\varepsilon]} h'(x)$. But

$$h'(x) = \left(e^{-x \ln c} \right)' = \ln(1/c) c^{-x}$$

and because $c < 1$ we have that $\max_{x \in [0, 1/\varepsilon]} h'(x) = c^{-1/\varepsilon} \ln(1/c)$.

Let now $\kappa(x_1, x_2) = -p(x_1)/\varepsilon$ and $\kappa(y_1, y_2) = -p(y_1)/\varepsilon$. Since $x_1 \neq x_2$ and $y_1 \neq y_2$ we have $d(x_1, x_2) = B(c^{-p(x_1)/\varepsilon})$ and $d(y_1, y_2) = B(c^{-p(y_1)/\varepsilon})$. Since $B(\kappa(x, y))$ is just an linear interpolation of points that belong to $c^{\kappa x, y}$ using the Mean Value Theorem we have that $|B(\kappa(x_1, x_2)) - B(\kappa(y_1, y_2))| \leq \max_{x \in [0, 1/\varepsilon]} h'(x) \left| \frac{p(x_1)}{\varepsilon} - \frac{p(y_1)}{\varepsilon} \right|$

$$\begin{aligned} |d(x_1, x_2) - d(y_1, y_2)| &= |B(\kappa(x_1, x_2)) - B(\kappa(y_1, y_2))| \leq \left(\max_{x \in [0, 1/\varepsilon]} h'(x) \right) \left| \frac{p(x_1)}{\varepsilon} - \frac{p(y_1)}{\varepsilon} \right| \\ &\Rightarrow |d(x_1, x_2) - d(y_1, y_2)| \leq c^{-1/\varepsilon} \frac{\ln(1/c)}{\varepsilon} |p(x_1) - p(y_1)| \end{aligned}$$

Now if $|p(x_1) - p(y_1)| > \lambda |x_1 - y_1|$ then x_1, y_1 satisfy (CO3) and we have a solution for CONTINUOUS LOCALOPT. So $|p(x_1) - p(y_1)| \leq \lambda |x_1 - y_1|$ and from the last inequality we have that

$$|d(x_1, x_2) - d(y_1, y_2)| \leq c^{-1/\varepsilon} \lambda \frac{\ln(1/c)}{\varepsilon} |x_1 - y_1|$$

But this contradicts with (Od) since $\lambda' = \max \left\{ \lambda, \left\lceil c^{-1/\varepsilon} \lambda \frac{\ln(1/c)}{\varepsilon} \right\rceil \right\}$.

Finally it is easy to see that the size of the arithmetic circuits that we used for this reduction is polynomial in the size of the input. The only function that needs for explanation is that of d and λ' . We start with the observation that both c, c^{-1} are given and have descriptions of size only linear in the description of ε , since ε is a rational constant. The difficult term in the description of d is the term $B(\kappa(x, y))$. For this, we need to bound the size of $\kappa x, y$, let this bound be A . Then we can have precomputed the possible digits of $\lceil \kappa(x, y) \rceil$ using $\log(A)$ arithmetic circuits. Finally a final circuit combines the digits in order to get $\lceil \kappa(x, y) \rceil$. Now to compute $c^{\lceil \kappa(x, y) \rceil}$ for each $m_i 2^i$ of the $\log(A)$ digits of $\kappa(x, y)$ we compute the corresponding power 2^i with repeated squaring using $O(i)$ arithmetic gates. Then we combine the results such to compute $c^{\lceil \kappa(x, y) \rceil}$. This whole process needs $O(\log^2 A)$ arithmetic gates. Since $A \leq 1/\varepsilon$ the overall circuit for d needs $\text{poly}(1/\varepsilon)$ arithmetic gates. For λ' we can also do a similar process but we have to bound $c^{-\kappa(x, y)}$. We can see that that $c^{-\kappa(x, y)} \leq c^{-1/\varepsilon} = (1 - 0.1\varepsilon)^{-1/\varepsilon} \leq e^{10}$. Therefore the size of $c^{-1/\varepsilon}$ is bounded its ceil can be computed using a polynomial sized circuit. \square

Proof of Theorem 3. Obviously because of Theorem 2, BANACH belongs to CLS.

For the opposite direction, we use the same reduction as in the proof of Theorem 2. We then prove that d satisfies the desired properties. We remind that we used the following instance of BANACH for the reduction

$$f' = f, d(x, y) = B(\kappa(x, y)) \cdot d_S(x, y), \varepsilon' = \frac{1}{\sqrt{c}}, \lambda' = \max \left\{ \lambda, \left\lceil c^{-1/\varepsilon} \lambda \frac{\ln(1/c)}{\varepsilon} \right\rceil \right\}, c = 1 - 0.1\varepsilon$$

We first prove that d is always a distance metric.

- (i) Obvious from the definition of d .
- (ii) If $x \neq y$ then $d_S(x, y) > 0$. Also always $c^{\kappa(x, y)} > 0$, therefore $d(x, y) > 0$. Now since $d_S(x, x) = 0$ we also have $d(x, x) = 0$.
- (iii) It is obvious from the definition of κ that $\kappa(x, y) = \kappa(y, x)$ and since d_S is a distance metric, the same is true for the d_S and thus for d .
- (iv) Without loss of generality we assume that $p(x) \geq p(y)$. We consider the following cases

$\mathbf{p}(\mathbf{x}) \geq \mathbf{p}(\mathbf{z})$ then we have $d(x, y) = d(x, z)$ and therefore obviously $d(x, y) \leq d(x, z) + d(z, y)$.

$\mathbf{p}(\mathbf{x}) \leq \mathbf{p}(\mathbf{z})$ then we have $d(x, y) = B\left(-\frac{p(x)}{2\varepsilon}\right)$, $d(x, z) = d(z, y) = B\left(-\frac{p(z)}{2\varepsilon}\right)$ but since $p(x) \leq p(z)$ obviously $2B\left(-\frac{p(z)}{2\varepsilon}\right) \geq B\left(-\frac{p(x)}{2\varepsilon}\right)$.

Finally we will show the completeness of $([0, 1]^3, d)$. We first observe that for all $x \neq y$, $d(x, y) > 1$, this comes from the fact that $c < 1$ and so $c^{-p(x)/\varepsilon} > 1$.

Now let (x_n) be a Cauchy sequence then $\forall \delta > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$, $d(x_n, x_m) \leq \delta$. We set $\delta = 1/2$ then there exists $N \in \mathbb{N}$ such that $\forall n, m > N$, $d(x_n, x_m) < 1/2$. But from the previous observation this implies $d(x_n, x_m) = 0$ and since d defines a metric we get $x_n = x_m$. Therefore (x_n) is constant for all $n > N$ and obviously converges. This means that every Cauchy sequence converges and so $([0, 1]^3, d)$ is a complete metric space. \square

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References

- [Bes59] C. Bessaga. On the converse of banach "fixed-point principle". *Colloquium Mathematicae*, 7(1):41–43, 1959.
- [BPR15] Nir Bitansky, Omer Paneth, and Alon Rosen. On the cryptographic hardness of finding a nash equilibrium. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*, pages 1480–1498. IEEE, 2015.
- [BTN01] Aharon Ben-Tal and Arkadi Nemirovski. *Lectures on modern convex optimization: analysis, algorithms, and engineering applications*. SIAM, 2001.
- [BV04] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.
- [Con14] Keith Conrad. The contraction mapping theorem. *Expository paper. University of Connecticut, College of Liberal Arts and Sciences, Department of Mathematics*, 2014.
- [DP11] Constantinos Daskalakis and Christos H. Papadimitriou. Continuous local search. In *Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2011, San Francisco, California, USA, January 23-25, 2011*, pages 790–804, 2011.
- [FGMS17] John Fearnley, Spencer Gordon, Ruta Mehta, and Rahul Savani. Cls: New problems and completeness. *arXiv preprint arXiv:1702.06017*, 2017.

- [Har14] Moritz Hardt. Understanding alternating minimization for matrix completion. In *Foundations of Computer Science (FOCS), 2014 IEEE 55th Annual Symposium on*, pages 651–660. IEEE, 2014.
- [HY17] Pavel Hubacek and Eylon Yogev. Hardness of continuous local search: Query complexity and cryptographic lower bounds. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1352–1371. SIAM, 2017.
- [JPY88] David S. Johnson, Christos H. Papadimitriou, and Mihalis Yannakakis. How easy is local search? *J. Comput. Syst. Sci.*, 37(1):79–100, 1988.
- [Kör10] T Körner. Metric and topological spaces, 2010.
- [LSJR16] Jason D. Lee, Max Simchowitz, Michael I. Jordan, and Benjamin Recht. Gradient descent only converges to minimizers. In *Proceedings of the 29th Conference on Learning Theory, COLT 2016, New York, USA, June 23-26, 2016*, pages 1246–1257, 2016.
- [Mey67] Philip R. Meyers. A converse to banach’s contraction theorem. *Journal of Research of the National Bureau of Standards Section B Mathematics and Mathematical Physics*, 71B(2 and 3):73, apr 1967.
- [Nes13] Yuri Nesterov. *Introductory lectures on convex optimization: A basic course*, volume 87. Springer Science & Business Media, 2013.
- [Pan11] Maysum Panju. Iterative methods for computing eigenvalues and eigenvectors. *arXiv preprint arXiv:1105.1185*, 2011.
- [Pap94] Christos H. Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *J. Comput. Syst. Sci.*, 48(3):498–532, 1994.
- [PP16] Ioannis Panageas and Georgios Piliouras. Gradient descent converges to minimizers: The case of non-isolated critical points. *CoRR*, abs/1605.00405, 2016.

A Preliminaries

A.1 Basic Definitions

Topological Spaces Let \mathcal{D} be a set and τ a collection of subsets of \mathcal{D} with the following properties.

- (a) The empty set $\emptyset \in \tau$ and the space $\mathcal{D} \in \tau$.
- (b) If $U_a \in \tau$ for all $a \in A$ then $\bigcup_{a \in A} U_a \in \tau$.
- (c) If $U_j \in \tau$ for all $1 \leq j \leq n \in \mathbb{N}$, then $\bigcap_{j=1}^n U_j \in \tau$.

Then we say that τ is a *topology* on \mathcal{D} and that (\mathcal{D}, τ) is a *topological space*. We call *open sets* the members of τ . Also a subset C of \mathcal{D} is called *closed* if $\mathcal{D} \setminus C$ is an open set, i.e. belongs to τ . Let (\mathcal{D}, τ) be a topological space and A a subset of \mathcal{D} . We write

$$\begin{aligned} \text{Int}(A) &= \bigcup \{U \in \tau \mid U \subseteq A\} \\ \text{Clos}(A) &= \bigcap \{U \text{ closed} \mid A \subseteq U\} \end{aligned}$$

and we call $\text{Clos}(A)$ the *closure* of A and $\text{Int}(A)$ the *interior* of A . We now give a basic lemma without proof. A proof can be found in [Kör10].

Lemma 2. (i) $\text{Int}(A) = \{x \in A \mid \exists U \in \tau \text{ with } x \in U \subseteq A\}$.
(ii) $\text{Clos}(A) = \{x \in \mathcal{D} \mid \forall U \in \tau \text{ with } x \in U, \text{ we have } U \cap A \neq \emptyset\}$.

Metric Spaces The *diameter* of a set $W \subseteq \mathcal{D}$ according to the metric d is defined as

$$\text{diam}_d[W] = \max_{x, y \in W} d(x, y)$$

A metric space (\mathcal{D}, d) is called *bounded* if $\text{diam}_d[\mathcal{D}]$ is finite. We define $d_S : \mathcal{D}^2 \rightarrow \mathbb{R}$ by

$$d_S(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

then d_S is called the *discrete metric* on \mathcal{D} .

Remark. It is very easy to see that discrete metric is indeed a metric, i.e. it satisfies the conditions (i)-(iv).

Let (\mathcal{D}, d) and (\mathcal{X}, d') be metric spaces. A function $f : \mathcal{D} \rightarrow \mathcal{X}$ is called *continuous* if, given $x \in \mathcal{D}$ and $\varepsilon > 0$, we can find a $\delta(x, \varepsilon) > 0$ such that

$$d'(f(x), f(y)) < \varepsilon \text{ whenever } d(x, y) < \delta(x, \varepsilon)$$

We say that a subset $E \subseteq \mathcal{D}$ is *open* in \mathcal{D} if, whenever $e \in E$, we can find a $\delta > 0$ (depending on e) such that

$$x \in E \text{ whenever } d(x, e) < \delta$$

The next lemma connects the definition of open sets according to some metric with the definition of open sets in a topological space.

Lemma 3. *If (\mathcal{D}, d) is a metric space, then the collection of open sets forms a topology.*

We define the *open ball* of radius r around x to be $B(x, r) = \{y \in \mathcal{D} \mid d(x, y) < r\}$.

Closed Sets for Metric Spaces Consider a sequence (x_n) in a metric space (\mathcal{D}, d) . If $x \in \mathcal{D}$ and, given $\varepsilon > 0$, we can find an integer $N \in \mathbb{N}_1$ (depending maybe on ε) such that

$$d(x_n, x) < \varepsilon \text{ for all } n \geq N$$

then we say that $x_n \rightarrow x$ as $n \rightarrow \infty$ and that x is the *limit* of the sequence (x_n) .

A set $G \subseteq \mathcal{D}$ is said to be *closed* if, whenever $x_n \in G$ and $x_n \rightarrow x$ then $x \in G$. A proof of the following lemma can be found in [Kör10].

Lemma 4. *Let (\mathcal{D}, d) be a metric space and A a subset of \mathcal{D} . Then $\text{Clos}(A)$ consists of all those $x \in \mathcal{D}$ such that we can find (x_n) with $x_n \in A$ with $d(x_n, x) \rightarrow 0$.*

We define the *closed ball* of radius r around x to be $\bar{B}(x, r) = \{y \in \mathcal{D} | d(x, y) \leq r\}$.

A subset G of a metric space (\mathcal{D}, d) is called *compact* if G is closed and every sequence in G has a convergent subsequence. A metric space (\mathcal{D}, d) is called compact if \mathcal{D} is compact, *locally compact* if for any $x \in \mathcal{D}$, x has a neighborhood that is compact and *proper* if every closed ball is compact.

Complete Metric Spaces We say that a sequence (x_n) in \mathcal{D} is *Cauchy sequence* (or *d-Cauchy sequence* if the distance metric is not clear from the context) if, given $\varepsilon > 0$, we can find $N(\varepsilon) \in \mathbb{N}_1$ with

$$d(x_n, x_m) < \varepsilon \text{ whenever } n, m \geq N(\varepsilon)$$

A metric space (\mathcal{D}, d) is *complete* if every Cauchy sequence converges.

Two metrics d, d' of the same set \mathcal{D} are called *topologically equivalent* (or just *equivalent*) if for every sequence (x_n) in \mathcal{D} , (x_n) is d -Cauchy sequence if and only if it is d' -Cauchy sequence.

Definition 5. *Let (\mathcal{D}, d) be a metric spaces. A function $f : \mathcal{D} \rightarrow \mathcal{D}$ is called continuous with respect to d , if given $x \in \mathcal{D}$ and $\varepsilon > 0$, we can find a $\delta(x, \varepsilon)$ such that*

$$d'(f(x), f(y)) < \varepsilon \text{ whenever } d(x, y) < \delta(x, \varepsilon)$$

Lipschitz Continuity Let (\mathcal{D}, d) and (\mathcal{X}, d') be metric spaces. A function $f : \mathcal{D} \rightarrow \mathcal{X}$ is *Lipschitz continuous* (or (d, d') -*Lipschitz continuous* if the distance metric is not clear from the context or *d-Lipschitz continuous* if $d = d'$) if there exists a positive constant $\lambda \in \mathbb{R}_+$ such that for all $x, y \in \mathcal{D}$

$$d'(f(x), f(y)) \leq \lambda d(x, y)$$

Lemma 5. *If a function $f : \mathcal{D} \rightarrow \mathcal{X}$ is Lipschitz continuous then it is continuous.*

Definition 6. *Let (\mathcal{D}, d) and (\mathcal{X}, d') be metric spaces. A function $f : \mathcal{D} \rightarrow \mathcal{X}$ is contraction (or (d, d') -contraction or d -contraction if $d = d'$) if there exists a positive constant $1 > c \in \mathbb{R}_+$ such that for all $x, y \in \mathcal{D}$*

$$d'(f(x), f(y)) \leq c d(x, y)$$

If $c = 1$ then we call f non-expansion.

A *fixed point* of a selfmap f is any point $x^* \in \mathcal{D}$ such that $f(x^*) = x^*$.

A.2 Introduction to Power Method

Let $A \in \mathbb{R}^{n \times n}$. Recall that if q is an eigenvector for A with eigenvalue λ , then $Aq = \lambda q$, and in general, $A^k q = \lambda^k q$ for all $k \in \mathbb{N}$. This observation is the foundation of the *power iteration method*.

Suppose that the set $\{q_i\}$ of unit eigenvectors of A forms a basis of \mathbb{R}^n , and has corresponding set of real eigenvalues $\{\lambda_i\}$ such that $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$. Let v_0 be an arbitrary initial vector, not perpendicular to q_1 , with $\|v_0\| = 1$. We can write v_0 as a linear combination of the eigenvectors of A for some $c_1, \dots, c_n \in \mathbb{R}$ we have that

$$v_0 = c_1 q_1 + c_2 q_2 + \dots + c_n q_n$$

and since we assumed that v_0 is not perpendicular to q_1 we have that $c_1 \neq 0$.

Also

$$Av_0 = c_1 \lambda_1 q_1 + c_2 \lambda_2 q_2 + \dots + c_n \lambda_n q_n$$

and therefore

$$\begin{aligned} Av_k &= c_1 \lambda_1^k q_1 + c_2 \lambda_2^k q_2 + \dots + c_n \lambda_n^k q_n \\ &= \lambda_1^k \left(c_1 q_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k q_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^k q_n \right) \end{aligned}$$

Since the eigenvalues are assumed to be real, distinct, and ordered by decreasing magnitude, it follows that

$$\lim_{k \rightarrow \infty} \left(\frac{\lambda_i}{\lambda_1} \right)^k = 0$$

So, as k increases, $A^k v_0$ approaches $c_1 \lambda_1^k q_1$, and thus for large values,

$$\frac{A^k v_0}{\|A^k v_0\|} \rightarrow q_1 \text{ as } k \rightarrow \infty$$

The power iteration method is simple and elegant, but suffers some drawbacks. Except from a measure 0 of initial conditions, the method returns a single eigenvector, corresponding to the eigenvalue of largest magnitude. In addition, convergence is only guaranteed if the eigenvalues are distinct—in particular, the two eigenvalues of largest absolute value must have distinct magnitudes. The rate of convergence primarily depends upon the ratio of these magnitudes, so if the two largest eigenvalues have similar sizes, then the convergence will be slow.

In spite of its drawbacks, the power method is still used in many applications, since it works well on large, sparse matrices when only a single eigenvector is needed. However, there are other methods overcoming some of the issues with the power iteration method.

B Proof of Theorem 1

The construction of d_c starts with an open neighborhood of x^* with some desired properties. In order to satisfy (2b), this open neighborhood W must have $\text{diam}_d[W] \leq \varepsilon$.

Lemma 6. *There exists an open neighborhood W of x^* such that*

$$f(W) \subseteq W \tag{3a}$$

$$\text{diam}_d[W] \leq \varepsilon \tag{3b}$$

Proof. From the hypothesis of the theorem there exists an open neighborhood U such that $f^{[n]}(U) \rightarrow \{x^*\}$. This implies that any open subset V of U satisfies $f^{[n]}(V) \rightarrow \{x^*\}$. Therefore we can choose a $V = \text{Int}(\bar{B}(x^*, \varepsilon))$ such that $\text{diam}_d[V] \leq \varepsilon$ and $f^{[n]}(V) \rightarrow \{x^*\}$. For simplicity of the notation we just assume refer to V as U and so $\text{diam}_d[U] \leq \varepsilon$.

Starting from U we prove the existence of W . For this, we will prove that there exists an open neighborhood W of x^* such that $f(W) \subset W$ and $W \subset U$. The latter implies $f^{[n]}(W) \rightarrow \{x^*\}$ and $\text{diam}_d[W] \leq \varepsilon$.

Since $f^{[n]}(U) \rightarrow \{x^*\}$, there is an integer k such that $f^{[k]}(U) \subseteq U$. Let

$$W = \bigcap_{j=0}^{k-1} f^{[-j]}(U) \subseteq U$$

Then for any $x \in W$ and for any $1 \leq j \leq k-1$ it holds that $x \in f^{[-j]}(U)$ and thus $f(x) \in f^{[-(j-1)]}(U)$. Moreover $x \in U$, so that $f^{[k]}(x) \in f^{[k]}(U) \subset U$ and thus $f(x) \in f^{[-(k-1)]}(U)$. Hence $x \in W$ implies $f(x) \in W$, which was to be shown. The diameter of W can be bounded by the diameter of U and hence is less than ε . \blacksquare

We now proceed to the main line of the proof. The construction follows three steps:

- I. We first construct a metric d_M , which is topologically equivalent to d , and with respect to which f is non-expanding. It also holds that $d_M(x, y) \geq d(x, y)$ for all $x, y \in \mathcal{D}$ and therefore Property (2b) can be transferred from d_M to d .
- II. Given d_M , we proceed to construct a “distance” function $\rho_{c, \varepsilon}$, which satisfies (2a) and all the metric properties except maybe for the triangle inequality. Moreover $\rho_{c, \varepsilon}$ satisfies that $\rho_{c, \varepsilon}(x, y) \geq d_M(x, y)$ if $\max\{d(x^*, x), d(x^*, y)\} \geq \varepsilon$, and therefore (2b) is preserved.
- III. Given $\rho_{c, \varepsilon}$, we construct the sought after metric $d_{c, \varepsilon}$ by taking it equal to the $\rho_{c, \varepsilon}$ -geodesic distance. Given the properties of $\rho_{c, \varepsilon}$ and the definition of $d_{c, \varepsilon}$, we can prove that $d_{c, \varepsilon}$ is a metric and Properties (2a) and (2b) hold.

I. Construction of d_M

We set

$$d_M(x, y) = \max_{n \in \mathbb{N}} \{d(f^{[n]}(x), f^{[n]}(y))\}$$

The fact that this maximum is finite can be proved using the condition 2. of the theorem. Indeed, since $d(f^{[n]}(x), x^*) \rightarrow 0$ and $d(f^{[n]}(y), x^*) \rightarrow 0$, for any $\delta > 0$ there is a number $N \in \mathbb{N}$ such that $d(f^{[n]}(x), x^*) \leq \delta$ and $d(f^{[n]}(y), x^*) \leq \delta$ for all $n > N$. Now if let $\delta = d(x, y)$ we get that $\max_{n \geq N} \{d(f^{[n]}(x), f^{[n]}(y))\} \leq d(x, y)$ and therefore $\max_{n \in \mathbb{N}} \{d(f^{[n]}(x), f^{[n]}(y))\} = \max_{0 \leq n \leq N} \{d(f^{[n]}(x), f^{[n]}(y))\}$. Hence the maximum has a finite value. Observe also that by definition it holds that

$$d_M(f(x), f(y)) \leq d_M(x, y)$$

and hence f is a non-expansion according to d_M . It only remains to prove that d_M satisfies the properties of a metric function. The positive definiteness and symmetry of d_M follow immediately from the corresponding properties of d . The fact that $d_M(x, y) \neq 0$ for $x \neq y$ follows from the fact that $d(x, y) \leq d_M(x, y)$, which follows directly from the definition of d_M since $f^{[0]}(x) = x$. It remains to prove the triangle inequality. For this we observe that by the definition of d_M and using the fact that the maximum in the definition of d_M for any $x, y \in \mathcal{D}$ is finite, there exists an $n \in \mathbb{N}$

such that

$$\begin{aligned} d_M(x, z) &= d(f^{[n]}(x), f^{[n]}(z)) \leq \\ &\leq d(f^{[n]}(x), f^{[n]}(y)) + d(f^{[n]}(y), f^{[n]}(z)) \leq \\ &\leq d_M(x, y) + d_M(y, z). \end{aligned}$$

Thus d_M is indeed a metric. We now show that d_M is topologically equivalent to d . From the inequality $d(x, y) \leq d_M(x, y)$ it follows that any d_M -convergent sequence is also d -convergent, with the same limit point. To prove the implication in the opposite direction, note that condition 2. of the hypotheses of the theorem implies the existence for each $\eta > 0$ of an N such that

$$\text{diam}_d [f^{[n]}(W)] < \eta \quad \text{for } n > N.$$

For each $x \in \mathcal{D}$, it follows from 2. that

$$\nu(x) = \min_{n \in \mathbb{N}, f^{[n]}(x) \in W} \{n\} \quad (4)$$

is finite. Since f is continuous, there is an $\delta > 0$ so small that $d(x, y) < \delta$ implies

$$f^{[\nu(x)]}(y) \in W \text{ and } d(f^{[j]}(x), f^{[j]}(y)) < \eta \text{ for } 0 \leq j \leq N + \nu(x). \quad (5)$$

By (3a) we have $f^{[n+N+\nu(x)]}(x) \in f^{[n+N]}(W)$ and $f^{[n+N+\nu(x)]}(y) \in f^{[n+N]}(W)$ for all $n > 0$, so that the (5) implies

$$d(f^{[j]}(x), f^{[j]}(y)) < \eta \quad \text{for } j > N + \nu(x).$$

Thus $d(x, y) \leq \delta$ implies $d_M(x, y) \leq \eta$. This shows that a sequence which is d -convergent to x is also d_M -convergent to x , completing the proof of topological equivalence. Finally since d and d_M are topologically equivalent and d is complete for \mathcal{D} it follows that d_M is also complete for \mathcal{D} .

II. Construction of ρ_c

We begin by defining K_n to be the closure of $f^n(W)$ for $n \geq 0$, and $K_{(-n)} = f^{[-n]}(K_0)$, so that $f^{[n]}(W) \rightarrow \{x^*\}$ implies

$$K_n \rightarrow \{x^*\} \quad \text{as } n \rightarrow \infty. \quad (6)$$

For $x \in K_0 \setminus \{x^*\}$, set

$$n(x) = \max_{x \in K_n} \{n\} \geq 0$$

finiteness is assured by (6). Let also $n(x^*) = \infty$, and for $x \in \mathcal{D} \setminus K_0$ set

$$n(x) = - \min_{f^{[m]}(x) \in K_0} \{m\} = \max_{x \in K_n} \{n\} < 0.$$

which must exist by condition 2. Letting also $\kappa(x, y) = \min\{n(x), n(y)\}$, we define ρ_c to be

$$\rho_c(x, y) = c^{\kappa(x, y)} d_M(x, y).$$

We can now prove that ρ_c satisfies all the distance metric requirements except maybe triangle inequality. Positive definiteness and symmetry is obvious. Also since d_M is a distance metric and $\kappa(x, y) \geq 0$ is finite at every point except from $\kappa(x^*, x^*)$, we get that $\rho_c(x, y) = 0 \Leftrightarrow x = y$. Now from the non-expansion property of f with respect to d_M and from the fact that $n(f(x)) \geq n(x) + 1$ we get that

$$\rho_c(f(x), f(y)) \leq c \cdot \rho_c(x, y) \quad (7)$$

and this concludes the proof of this step.

III. Construction of d_c

In this last step what we do is that we assign the distance between two points to be the length of the shortest path that connects these two points, with the lengths computed according to ρ_c . Then the distance satisfies the triangle inequality because of the shortest path property.

Formally, denote by S_{xy} the set of chains $s_{xy} = (x = x_0, x_1, \dots, x_m = y)$ from x to y with associated lengths $L_c(s_{xy}) = \sum_{i=1}^m \rho_c(x_i, x_{i-1})$. We define

$$d_c(x, y) = \inf\{L_c(s_{xy}) \mid s_{xy} \in S_{xy}\}. \quad (8)$$

We will prove that d_c is the desired metric. That f is a contraction with constant c with respect to d_c follows by applying (7) to the links $[x_{i-1}, x_i]$ of any chain s_{xy} . Clearly d_c is symmetric and $d_c(x, x) = 0$. The triangle law holds since following a s_{xy} with a s_{yz} yields a s_{xz} . It remains to show that it is positive definite.

Consider any $x \neq x^*$ and $y \neq x$ and assume $n(x) \leq n(y)$ without loss of generality. If $y \neq x^*$, any chain s_{xy} either lies in $\mathcal{D} \setminus K_{n(y)+1}$, or has a last link which leaves $K_{n(y)+1}$, so that

$$d_c(x, y) \geq c^{n(y)} \min\{d_M(x, y), d_M(x, K_{n(y)+1})\} > 0. \quad (9)$$

The remaining case, $y = x^*$ is covered by

$$d_c(x, y) \geq c^{n(x)} d_M(x, K_{n(x)+1}) > 0. \quad (10)$$

Thus d_c is a distance metric. We now have to prove that d_c is equivalent to d_M . Let $B_\nu = \mathcal{D} \setminus f^{[-\nu]}(W)$ for $\nu \geq 0$, so that the definition of $\nu(x)$ (4) implies $d_M(x, B_{\nu(x)}) > 0$ and $n(x) \geq -\nu(x)$. For any $x \neq x^*$, if y obeys

$$d_M(x, y) < \delta(x) = \min\{d_M(x, K_{n(x)+1}), d_M(x, B_{\nu(x)})\} \quad (11)$$

then $n(x) \geq -\nu(x)$, so that (8) and (9), the last with x and y interchanged, imply

$$c^{n(x)} d_M(x, y) \leq d_c(x, y) \leq \rho_c(x, y) \leq c^{-\nu(x)} d_M(x, y). \quad (12)$$

Now choose $k(x) > \max\{0, n(x)\}$ such that $z \in K_{k(x)}$ implies $d_M(z, x^*) < d_c(x, x^*)/2$. Then $d_c(x, K_{k(x)}) \geq d_c(x, x^*)/2$, so that if y obeys

$$d_c(x, y) < d_c(x, x^*)/2 \quad (13)$$

then only chains disjoint from $K_{k(x)}$ need enter (8), implying

$$d_c(x, y) \geq c^{k(x)} d_M(x, y). \quad (14)$$

In particular, if

$$d_c(x, y) < \min\{d_c(x, x^*)/2, c^{k(x)} \delta(x)\}$$

then with (13) and (14) this implies (11) and hence (12) applies. Thus $d_c(x_n, x) \rightarrow 0$ whenever $d_M(x_n, x) \rightarrow 0$.

Now if $x = x^*$, note first that if $d_M(x^*, y) < d_M(x^*, B_0)$, then

$$d_c(x^*, y) \leq \rho_c(x^*, y) \leq d_M(x^*, y). \quad (15)$$

Also note that for any $\eta > 0$, $f^{[n]}(W) \rightarrow \{x^*\}$ guarantees an $N(\eta) > 0$ such that $d_M(x^*, z) < \eta/2$ for all $z \in K_{N(\eta)}$. Then $d_M(x^*, y) > \eta$ implies that $d_M(y, K_{N(\eta)}) \geq \eta/2$ and thus that

$$d_c(x^*, y) \geq d_c(K_{N(\eta)}, y) \geq c^{N(\eta)} \eta/2$$

Hence $d_c(x_n, x^*) \rightarrow 0$ if and only if $d_M(x_n, x^*) \rightarrow 0$.

To show that d_M -completeness is preserved, assume that (x_n) is a d_c -Cauchy sequence and that (X, d_M) is complete. If (x_n) does not converge to x^* then since d_c and d_M are equivalent, for some $N \in \mathbb{N}$ and all sufficiently large n , $n(x_n) < N$.

Now exactly as above choose $k((x_n)) = P > \max 0, N$ such that $z \in K_{k((x_n))}$ implies

$$d_M(x^*, z) < \inf_{i \in \mathbb{N}} \left\{ \frac{d_c(x_i, x^*)}{2} \right\} = \frac{R}{2}$$

then since (x_n) is a Cauchy sequence there is an $i \in \mathbb{N}_1$ such that

$$d_c(x_p, x_{p+j}) < \frac{R}{2}$$

for all $p > i$, and using (14) with $k(x) = P$, we have

$$c^{-P} d_c(x_p, x_{p+j}) \geq d_M(x_p, x_{p+j})$$

so that (x_n) is a d_M -Cauchy sequence. Therefore since (\mathcal{D}, d_M) is complete, the topological space (\mathcal{D}, d_c) is complete too.

The final step is to prove (2b). Let $A = \text{diam}_d [\bar{B}(x^*, 2\varepsilon)]$ and without loss of generality $d(x, x^*) \geq d(y, x^*)$.

If either $d_M(x, x^*) \leq \varepsilon$ or $d_M(y, x^*) \leq \varepsilon$ then we are done since as we have seen in the construction of d_M , $d_M(x, y) \geq d(x, y)$, thus either $d(x, x^*) \leq \varepsilon$ or $d(y, x^*) \leq \varepsilon$ and (2b) is satisfied. So we may assume that $d(x, x^*) \geq \varepsilon$ and $d(y, x^*) \geq \varepsilon$. Therefore $x, y \in \mathcal{D} \setminus K_0$ and which translates to $n(x), n(y) < 0$. So using the same argument as when we derived (9) but with K_0 instead of $K_{n(y)+1}$ we get

$$d_c(x, y) \geq \min\{d_M(x, y), d_M(x, K_0)\}. \quad (16)$$

Now we consider two cases according to the value of $d_M(x, K_0)$. If $d_M(x, K_0) \geq \varepsilon$ then

$$d_c(x, y) \leq \varepsilon \implies d_M(x, y) \leq \varepsilon \leq A \implies d(x, y) \leq \varepsilon.$$

Otherwise if $d_M(x, K_0) \leq \varepsilon$ then $d(x, K_0) \leq \varepsilon$ and by triangle inequality $d(x, x^*) \leq 2\varepsilon$. By our assumption for the relative position of x and y we also get $d(y, x^*) \leq 2\varepsilon$ and therefore $x, y \in \bar{B}(x^*, 2\varepsilon)$. Thus, $d(x, y) \leq \text{diam}_d [\bar{B}(x^*, 2\varepsilon)]$. \square